**This document is a correction to Section 5-3.2 in Montgomery D. C. "Introduction to Statistical Quality Control", 4th Edition, 2001. The main error in the text is that Eq. 5-31 is not an unbiased estimator of E(S); the right-hand side of Eq. 5-31 is an unbiased estimator of $\sigma$. In addition, there is rounding error in the calculation of "$\bar{S}$".**

***Noel Artiles-León, Ph.D., Jan. 2001***

## 5-3.2 The $\bar{X}$ and S Control Charts with Variable Sample Size

The $\bar{x}$ and S control charts are relatively easy to apply in cases where the sample sizes are variable. In this case, we should use a weighted average approach in estimating $\mu$ and $\sigma$. If $n_i$ is the number of observations in the ith sample, then use

$$\hat{\mu} = \frac{\sum_{i=1}^{m} n_i x_i}{\sum_{i=1}^{m} n_i}, \quad (\mathbf{5\text{-}30}), \quad \text{and} \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^{m} (n_i - 1) S_i^2}{\sum_{i=1}^{m} n_i - m}, \quad (\mathbf{5\text{-}31}),$$

to estimate the process mean and variance, respectively. Notice that, since $S_i^2$ is an unbiased estimator of $\sigma^2$, Ec. 5-31, being a weighted average of unbiased estimators of $\sigma^2$, gives an unbiased estimator of $\sigma^2$. The control limits are then calculated as:

For $\bar{X}$ chart:

$$\hat{\mu} \pm 3\sqrt{\frac{\hat{\sigma}^2}{n_i}}$$

For S chart: $\hat{\sigma}(c_4 \pm 3\sqrt{1 - c_4^2})$, or LCL = $B_5\hat{\sigma}$ and UCL = $B_6\hat{\sigma}$

The constants $c_4$, $B_5$ and $B_6$ will depend on the sample size used in each individual subgroup.

For $S^2$ chart:

$$\hat{\sigma}^2 \left(1 \pm 3\sqrt{\frac{2}{n_i - 1}}\right)$$

**EXAMPLE 5-4**

Consider the data in Table 5-4, which is a modification of the piston-ring data used previously. Note that the sample sizes vary from n=3 to n=5. We may use the procedure described above to set up the $\bar{X}$ and S control charts. The estimators of $\mu$ and $\sigma$, using Ecs. 5-30 and 5-31, are:

$\hat{\mu}$ = 8362.09 / 113 = 74.0008 and

$\hat{\sigma}^2$ = (0.00021820x4 + 0.00002100x2 + 0.00021750x4 + ... + 0.00026170x4) / 88 =

$\hat{\sigma}^2$ = 0.009319933/88 = 0.000105908, or $\hat{\sigma}$ = 0.01029

Therefore, the center line of the $\bar{X}$ chart = 74.0008, and the center line of the S chart is:

- for n = 3, $E(S) = c_4\,\sigma = 0.8862\,(.01029) = 0.009120$
- for n = 4, $E(S) = c_4\,\sigma = 0.9213\,(.01029) = 0.009481$
- for n = 5, $E(S) = c_4\,\sigma = 0.9400\,(.01029) = 0.009674$

The control limits for the $\bar{X}$ chart are: $\hat{\mu} \pm 3\sqrt{\frac{\hat{\sigma}^2}{n_i}}$,

- for n = 3, $74.0008 \pm 3(0.01029)/\sqrt{3} = (73.983, 74.019)$
- for n = 4, $74.0008 \pm 3(0.01029)/\sqrt{4} = (73.985, 74.016)$
- for n = 5, $74.0008 \pm 3(0.01029)/\sqrt{5} = (73.987, 74.015)$

The control limits for the S chart are: $\hat{\sigma}(c_4 \pm 3\sqrt{1-c_4^2})$

- for n = 3, $(0.01029)[0.8862+3\sqrt{(1+0.8862^2)}] = 0.0234$
- for n = 4, $(0.01029)[0.9213+3\sqrt{(1+0.9213^2)}] = 0.0215$
- for n = 5, $(0.01029)[0.9400+3\sqrt{(1+0.9400^2)}] = 0.0202$

The control limits calculations for all 25 samples are summarized in Table 5-5. The control charts are plotted in Fig. 5-20.

| k | Observations | | | | | $\bar{X}$ | $S^2$ | S | $\bar{X}$ chart | | S chart |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | LCL | UCL | UCL |
| 1 | 74.030 | 74.002 | 74.019 | 73.992 | 74.008 | 74.0102 | 0.00021820 | 0.0148 | 73.987 | 74.015 | 0.0202 |
| 2 | 73.995 | 73.992 | 74.001 | | | 73.996 | 0.00002100 | 0.0046 | 73.983 | 74.019 | 0.0234 |
| 3 | 73.988 | 74.024 | 74.021 | 74.005 | 74.002 | 74.0080 | 0.00021750 | 0.0147 | 73.987 | 74.015 | 0.0202 |
| 4 | 74.002 | 73.996 | 73.993 | 74.015 | 74.009 | 74.0030 | 0.00008250 | 0.0091 | 73.987 | 74.015 | 0.0202 |
| 5 | 73.992 | 74.007 | 74.015 | 73.989 | 74.014 | 74.0034 | 0.00014930 | 0.0122 | 73.987 | 74.015 | 0.0202 |
| 6 | 74.009 | 73.994 | 73.997 | 73.985 | | 73.9963 | 0.00009825 | 0.0099 | 73.985 | 74.016 | 0.0215 |
| 7 | 73.995 | 74.006 | 73.994 | 74.000 | | 73.9988 | 0.00003025 | 0.0055 | 73.985 | 74.016 | 0.0215 |
| 8 | 73.985 | 74.003 | 73.993 | 74.015 | 73.988 | 73.9968 | 0.00015020 | 0.0123 | 73.987 | 74.015 | 0.0202 |
| 9 | 74.008 | 73.995 | 74.009 | 74.005 | | 74.0043 | 0.00004092 | 0.0064 | 73.985 | 74.016 | 0.0215 |
| 10 | 73.998 | 74.000 | 73.990 | 74.007 | 73.995 | 73.9980 | 0.00003950 | 0.0063 | 73.987 | 74.015 | 0.0202 |
| 11 | 73.994 | 73.998 | 73.994 | 73.995 | 73.990 | 73.9942 | 0.00000820 | 0.0029 | 73.987 | 74.015 | 0.0202 |
| 12 | 74.004 | 74.000 | 74.007 | 74.000 | 73.996 | 74.0014 | 0.00001780 | 0.0042 | 73.987 | 74.015 | 0.0202 |
| 13 | 73.983 | 74.002 | 73.998 | | | 73.9943 | 0.00010033 | 0.0100 | 73.983 | 74.019 | 0.0234 |
| 14 | 74.006 | 73.967 | 73.994 | 74.000 | 73.984 | 73.9902 | 0.00023420 | 0.0153 | 73.987 | 74.015 | 0.0202 |
| 15 | 74.012 | 74.014 | 73.998 | | | 74.0080 | 0.00007600 | 0.0087 | 73.983 | 74.019 | 0.0234 |
| 16 | 74.000 | 73.984 | 74.005 | 73.998 | 73.996 | 73.9966 | 0.00006080 | 0.0078 | 73.987 | 74.015 | 0.0202 |
| 17 | 73.994 | 74.012 | 73.986 | 74.005 | | 73.9993 | 0.00013292 | 0.0115 | 73.985 | 74.016 | 0.0215 |
| 18 | 74.006 | 74.010 | 74.018 | 74.003 | 74.000 | 74.0074 | 0.00004880 | 0.0070 | 73.987 | 74.015 | 0.0202 |
| 19 | 73.984 | 74.002 | 74.003 | 74.005 | 73.997 | 73.9982 | 0.00007170 | 0.0085 | 73.987 | 74.015 | 0.0202 |
| 20 | 74.000 | 74.010 | 74.013 | | | 74.0077 | 0.00004633 | 0.0068 | 73.983 | 74.019 | 0.0234 |
| 21 | 73.982 | 74.001 | 74.015 | 74.005 | 73.996 | 73.9998 | 0.00014770 | 0.0122 | 73.987 | 74.015 | 0.0202 |
| 22 | 74.004 | 73.999 | 73.990 | 74.006 | 74.009 | 74.0016 | 0.00005530 | 0.0074 | 73.987 | 74.015 | 0.0202 |
| 23 | 74.010 | 73.989 | 73.990 | 74.009 | 74.014 | 74.0024 | 0.00014230 | 0.0119 | 73.987 | 74.015 | 0.0202 |
| 24 | 74.015 | 74.008 | 73.993 | 74.000 | 74.010 | 74.0052 | 0.00007570 | 0.0087 | 73.987 | 74.015 | 0.0202 |
| 25 | 73.982 | 73.984 | 73.995 | 74.017 | 74.013 | 73.9982 | 0.00026170 | 0.0162 | 73.987 | 74.015 | 0.0202 |